Student Name______________
Number__________________

ASCHAM SCHOOL

2024

YEAR 12

TRIAL

EXAMINATION

Mathematics

Extension 2

Total marks – 100

- Attempt Sections A and B.
- Section A is worth 10 marks.
- Recommended time on Section A: 15 minutes
- Answer Section A on the multiple choice answer sheet.
- Detach the multiple choice answer sheet from the back of the examination paper.
- Section B contains 6 questions worth 15 marks each.
- Recommended time on Section B: 2 hours 45 minutes
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher.

General Instructions

- Reading time – 10 minutes.
- Working time – 3 hours.
- Write using black non-erasable pen.
- NESA-approved calculators may be used.
- A NESA Reference Sheet is provided.
- All necessary working should be shown in every question.

SECTION A – 10 MULTIPLE CHOICE QUESTIONS 10 MARKS

ANSWER ON THE ANSWER SHEET

1 Which of the following does $\underset{\sim}{u} = \begin{pmatrix} 4t \\ 5t^2 \\ 3t^3 \end{pmatrix}$ describe?

A A circle
B A parabola
C A sphere
D None of the above

2 What are the roots of $x^3 - 7x^2 + 17x - 15 = 0$?

A $2+i, 2-i, 3$

B $-2+i, -2-i, 3$

C $2+i, 2-i, -3$

D $-2+i, -2-i, -3$

3 Let $z^n = \left(r\left(\cos\alpha + i\sin\alpha\right)\right)^n$. Which of the following is true?

A $\forall r, n \in \mathbb{N}, \exists x, y \in \mathbb{Q} : z^n = x + iy$

B $\forall x, y \in \mathbb{R}, \exists r, n \in \mathbb{N} : z^n = x + iy$

C $\forall r, n \in \mathbb{N}, \exists x, y \in \mathbb{R} : z^n = x + iy$

D $\forall x, y \in \mathbb{Z}, \exists r, n \in \mathbb{Z} : z^n = x + iy$

4 It is known that ω is a complex cube root of unity. What is the value of $\left(\omega^2+\omega-1\right)\left(\omega^2-\omega+1\right)$?

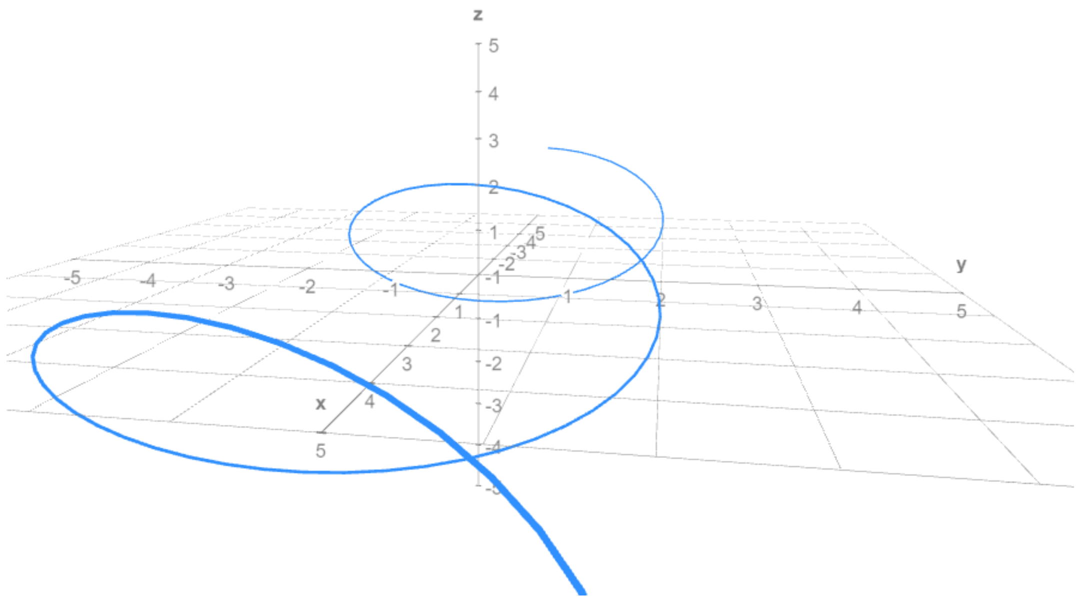
A ω

B 2ω

C 3ω

D 4ω

5 What is a possible equation for the graph below?



A $\underset{\sim}{u}=\begin{pmatrix}2\sin t\\ t\\ 2\cos t\end{pmatrix}$

B $\underset{\sim}{u}=\begin{pmatrix}t\\ 2\sin t\\ 2\cos t\end{pmatrix}$

C $\underset{\sim}{u}=\begin{pmatrix}2\cos t\\ t\\ 2\sin t\end{pmatrix}$

D $\underset{\sim}{u}=\begin{pmatrix}t\\ 2\cos t\\ 2\sin t\end{pmatrix}$

6 Which of the following is the converse of the following statement?
If you vape, then your lungs do not function properly.

A If you vape, your lungs function properly.

B If your lungs do not function properly, then you vape.

C If your lungs function properly, then you vape.

D If you do not vape, then your lungs do not function properly.

7 What is the modulus of the expression $\left(\dfrac{1+i\sqrt{3}}{1-i\sqrt{2}}\right)^{10}$?

A $\dfrac{4^{10}}{3^{10}}$

B $\dfrac{4^{10}}{3^{5}}$

C $\dfrac{2^{10}}{3^{5}}$

D $\dfrac{2^{10}}{3^{10}}$

8 Without calculating, what is the value of the integral $\int_0^{\pi} \sin^8 x\,dx$?

A $\dfrac{3\pi}{90}-1$

B $\dfrac{35\pi}{128}$

C $\dfrac{\pi}{2}$

D π

9 Consider the box of mass 2kg being held by two strings with tensions T_1 *and* T_2 respectively. The angles of depression the two strings make with the horizontal are θ *and* α respectively. Take acceleration due to gravity to be g.

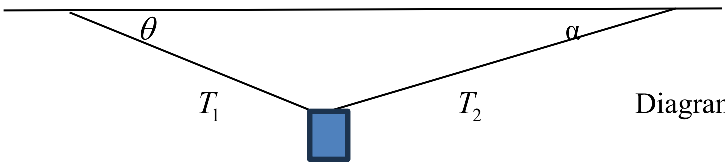


Diagram not to scale

What is the correct equation describing the vertical force on the object?

A $T_1 \sin\theta + T_2 \sin\alpha = 2g$

B $T_1 \sin\theta - T_2 \sin\alpha = 2g$

C $T_1 \cos\theta + T_2 \cos\alpha = 2g$

D $T_1 \cos\theta - T_2 \cos\alpha = 2g$

10 A particle P with unit mass is moving in the horizontal direction with an acceleration of $\ddot{x} = k^2 (x - A)$.

What is the correct equation for displacement as a function of t?

A $x = A\left(1 - e^{-kt}\right)$

B $x = A\left(1 + e^{-kt}\right)$

C $x = A\left(1 + e^{kt}\right)$

D $x = A\left(1 - e^{kt}\right)$

SECTION 2 – 6 QUESTIONS EACH WORTH 15 MARKS

Question 11 – Begin a new writing booklet

a Given $z = -1 + i\sqrt{3}$,

i find $\arg z$ *and* $\operatorname{mod} z$. **2**

ii find z^{15} in Cartesian form. **2**

b Find $\int_{-3}^{3} (x+4)\sqrt{9-x^2}\ dx$. **4**

c Consider the true statement:

$\forall a, b, c \in \mathbb{N}$: If either a or b are divisible by c, and a and b have no common factors, then the product ab is also divisible by c.

i Explain briefly why $n(n+1)(n-1)$ is divisible by 6 for $n \in \mathbb{N}$. **2**

ii Use proof by contradiction to prove that $n(n+1)(n-1)$ is not divisible by 30 if n is of the form $n = 5m \pm 2$ for some integer m. **3**

d Write the negation of the statement: **2**

There exists a student who studies for her exams with an ATAR < 90.

Question 12 – Begin a new writing booklet

a Find $\int \tan x \ln(\cos x)\, dx$. **2**

b Explain geometrically why $\int_1^e \ln x\, dx + \int_0^1 e^x\, dx = e$. **2**

c Let $z = \cos\theta + i\sin\theta$.

i Find the six roots of $z^6 = -1$ and plot them on an Argand diagram. **2**

ii Hence, or otherwise, find the four roots of $z^4 - z^2 + 1 = 0$. **2**

d The 4 complex numbers z_1, z_2, z_3, z_4 are represented by the points Z_1, Z_2, Z_3, Z_4 on an Argand diagram. In the diagram $Z_1Z_2Z_3Z_4$ is a rectangle where the dimensions are such that $2Z_1Z_2 = Z_2Z_3$.

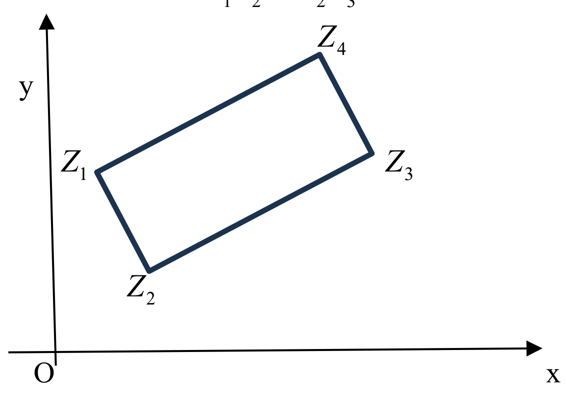


Diagram not to scale

Let $z_2 = \sqrt{3} + i$ *and* $z_3 = 4\sqrt{3} + 4i$.

i Find the complex number z_1 in Cartesian form. **2**

ii Find the complex number z_4 in Cartesian form. **2**

iii Prove that $\arg\left(\dfrac{z_4 - z_2}{z_3 - z_1}\right) = \tan^{-1}\dfrac{4}{3}$. **3**

Question 13 – Begin a new writing booklet

a Given that $a > 0, b > 0, s > 0$ *and* $a + b = s$,

i Prove that $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{s}$. **2**

ii Prove that $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{s^2}$. **2**

b The point $P(3,2,1)$ lies on the sphere S_1 with centre $C(2,3,4)$. The point Q is located such that PQ is a diameter.

i Find $Q(a,b,c)$. **1**

ii Show that the line with vector equation $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$ is a tangent to the sphere, S_1. **2**

iii Find the vector equation of the line l through Q which is perpendicular to both PQ and the direction of $\underset{\sim}{r}$. **3**

Question 13 continues on the next page…

Question 13 continued

c Consider the parallelogram $ABCD$ shown. The point F lies on BC such that $\overrightarrow{BF} = \frac{1}{3}\overrightarrow{BC}$ and the point E lies on DC such that $\overrightarrow{DE} = \frac{3}{4}\overrightarrow{DC}$. The point X is the intersection of $\overrightarrow{DF}$ *and* $\overrightarrow{AE}$. Let $\overrightarrow{DX} = \lambda\overrightarrow{DF}$ *and* $\overrightarrow{AX} = \mu\overrightarrow{AE}$ where λ and μ are parameters. Let $\overrightarrow{AB} = \underset{\sim}{p}$ *and* $\overrightarrow{AD} = \underset{\sim}{q}$.

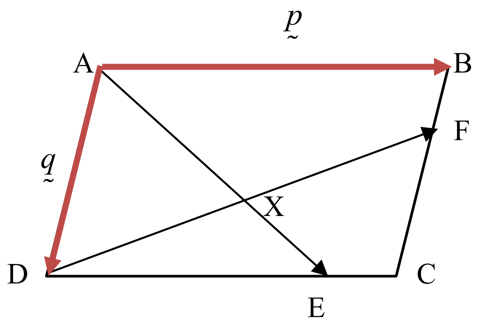


Diagram not to scale.

Copy the diagram.

i Prove that $\overrightarrow{AX} = \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right)$. **2**

ii Hence, or otherwise, prove that $\overrightarrow{AE}$ bisects DF. **3**

Question 14 – Begin a new writing booklet

a **i** Find the minimum value of $f(x) = x - n\ln x \quad for \quad x > 0, n = 1,2,3,4,\dots$ **2**

ii Hence prove that $\dfrac{e^x}{x^n} > \dfrac{e^n}{n^n}, \forall x > n$. **2**

b Let $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx \quad for \quad n = 1,2,3,\dots$.

i Show that $I_n = \dfrac{1}{n}\left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + \dfrac{n-1}{n} I_{n-2}$. **3**

ii Use the identity above to find $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$. **2**

c **i** Use mathematical induction to prove that $n! > 2^n \quad for \quad n > 3$. **3**

ii Hence prove that **2**

$$\sum_4^n \frac{1}{k!} < \sum_4^n \frac{1}{2^k}.$$

iii Explain why $\lim\limits_{n\to\infty} \left(\dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!} + \dfrac{1}{7!} + \dots + \dfrac{1}{n!}\right) < \dfrac{1}{8}$. **1**

Question 15 – Begin a new writing booklet

a **i** Prove that $\tan^{-1}(k+1) - \tan^{-1} k = \tan^{-1}\left(\dfrac{1}{k(k+1)+1}\right)$ *for* $k = 1, 2, 3, \ldots$ **2**

ii Hence prove that **3**

$$\lim_{n\to\infty}\left[\tan^{-1}\left(\frac{1}{1.2+1}\right)+\tan^{-1}\left(\frac{1}{2.3+1}\right)+\tan^{-1}\left(\frac{1}{3.4+1}\right)+\ldots+\tan^{-1}\left(\frac{1}{n(n+1)+1}\right)\right]=\frac{\pi}{4}$$

b A box jellyfish is floating on the ocean surface. At 11am, it is 3m below the wharf. At 1pm, high tide occurs and it is 1m below the wharf. Low tide occurs at 7pm. Assume it is moving in simple harmonic motion.

i Find the period. **2**

ii Find the amplitude. **3**

iii Find the first time after 11am that the tide is rising fastest. **2**

c By expanding $(1+i)^n$ in two different ways, show that **3**

$$1-\binom{n}{2}+\binom{n}{4}-\binom{n}{6}+\ldots=2^{\frac{n}{2}}\cos\frac{n\pi}{4}$$

Question 16 – Begin a new writing booklet

a Find $\int \dfrac{\cos x}{1+\cos x}\,dx$. **3**

b **i** Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$ using any method. **3**

ii Hence solve $16x^4 - 16x^2 + 1 = 0$. **2**

iii Hence, or otherwise, find the exact values of $\cos\left(\dfrac{\pi}{12}\right)$ and $\cos\left(\dfrac{5\pi}{12}\right)$. **2**

Question 16 continues on the next page…

Question 16 continued…

c A particle with acceleration $\ddot{x}$, displacement x and velocity v at time t moves such that $x = 2v - \tan^{-1} v - 2 + \dfrac{\pi}{4}$.

Initially the particle is at the origin and the velocity is 1.

i Show that $v\dfrac{dv}{dx} = \dfrac{v\left(1+v^2\right)}{1+2v^2}$. **2**

ii Find the time taken for the particle to reach a velocity of 7. **3**

The end! ☺

Solutions to
Title: Ascham 2024 Trial Math Ext 2 CORRECTED

MC

1. $\underset{\sim}{u} = \begin{pmatrix} 4t \\ 5t^2 \\ 3t^3 \end{pmatrix}$ is some sort of surface or curve

$t = \frac{x}{4} \quad y = 5\left(\frac{x^2}{16}\right) \quad z = 3\left(\frac{x^3}{64}\right)$

None of the above D.

2. Sum of roots $= \frac{-b}{a} = \frac{7}{1} = 7$

$2 + \cancel{i} + 2 - \cancel{i} + 3 = 7$ A

3. $z^n = r^n(i\sin n\alpha + \cos n\alpha)$

$\sin n\alpha$ & $\cos n\alpha$ might not be rational. C

4. $(\omega^2 + \omega - 1)(\omega^2 - \omega + 1)$

We know if $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$

then $(-1-1)(-\omega-\omega) = -2 \times -2\omega$
$= 4\omega$ D

5. Spiralling in y-z plane along x-axis. Point $(0, 0, 2)$.

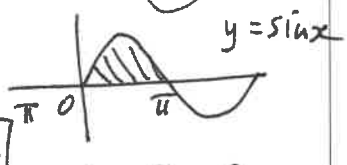
B

6. $P \Rightarrow Q$: Converse is $Q \Rightarrow P$.
$\therefore$ If your lungs don't function properly then you vape. B

7. $\left|\frac{z_1}{z_2}\right|^n = \frac{|z_1|^n}{|z_2|^n} \quad \therefore \left|\frac{1+i\sqrt{3}}{1-i\sqrt{2}}\right|^{10} = \frac{2^{10}}{(\sqrt{3})^{10}}$

C

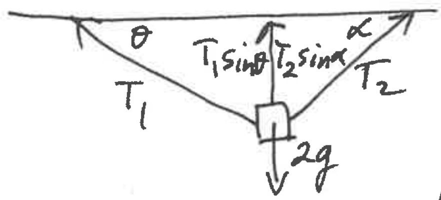
8. $\int_0^{\pi} \sin^8 x \, dx$



$\int_0^{\pi} \sin x \, dx = -\left[\cos x\right]_0^{\pi}$
$= -(-1) + (1)$
$= 2$
$< \pi$

B

9.



$T_1 \sin\theta + T_2 \sin\alpha = 2g$ A

10. $x = A(1 - e^{-kt}) = A - Ae^{-kt}$
$\dot{x} = 0 + kAe^{-kt}$
$\ddot{x} = -k^2 Ae^{-kt}$
$= -k^2(A - x)$
$= k^2(x - A)$

A

APOLOGIES FOR UNINTENDED ILLEGIBILITY.

Solutions to
Title: Ascham 2024 Trial Math Ext 2

Q 11

a) $z = -1 + i\sqrt{3}$

$= 2\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$

i) $\therefore\ |z| = 2 \quad \arg z = \frac{2\pi}{3}$. 2

ii) $z^{15} = 2^{15}\left(\cos\left(\frac{2\pi}{3}\times 15\right) + i\sin\left(\frac{2\pi}{3}\times 15\right)\right)$

$= 2^{15}(\cos 10\pi + i\sin 10\pi)$

$= 2^{15}(1 + 0i)$ 2

$= 2^{15} + 0i$

b) $\int_{-3}^{3} (x+4)\sqrt{9-x^2}\,dx$

$= \int_{-3}^{3} x\sqrt{9-x^2}\,dx + 4\int_{-3}^{3}\sqrt{9-x^2}\,dx$ ← OR $= 0$ since ODD.

$= -\frac{1}{2}\int_{-3}^{3} -2x\sqrt{9-x^2}\,dx + 4\left(\frac{\pi r^2}{2}\right)$

$= -\frac{1}{\cancel{2}}\left[(9-x^2)^{\frac{3}{2}} \times \frac{\cancel{2}}{3}\right]_{-3}^{3} + 4\left(\frac{\pi(3)^2}{\cancel{2}}\right)$

$= -\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}\left[\left(\sqrt{9-x^2}\right)^3\right]_{-3}^{3} + 18\pi$

$= -\frac{2}{3}[0 - 0] + 18\pi$ 4

$= 18\pi$

c) i) $n(n+1)(n-1)$ if $n \in \mathbb{N}$

Say n is even then $n+1$ or n or $n-1$ will be divisible by 3 so $n(n+1)(n-1)$ divisible by 6

OR Say n is odd then n is divisible by 3 or $n+1$ or $n-1$ is divisible by 3 and even so $n(n+1)(n-1)$ is divisible by 6. 2

c) Cont'd ii) RTP: $n(n+1)(n-1)$ is not divisible by 30 if $n = 5m \pm 2, m \in \mathbb{Z}$.

Proof: Assume $n = 5m \pm 2$, $m \in \mathbb{Z}$ and $\therefore n(n+1)(n-1)$ is divisible by 30.

Using the above statements if a or/and b are divisible by 5 then ab will be divisible by 5.

We already know $n(n+1)(n-1)$ is divisible by 6 so need to show divisible by 5 as well.

Assume $n = 5m + 2$:

$n(n+1)(n-1) = (5m+2)(5m+3)(5m+1)$

$= (5m+2)(25m^2 + 20m + 3)$

$= (5m+2)(5X + 3) \quad X \in \mathbb{Z}$

$= \underbrace{25mX + 15M + 10X}\ + 3$ 3

$= 5Y + 3$ for some $Y \in \mathbb{Z}$

$\neq 5Q$, $Q \in \mathbb{Z}$ $\therefore$ Not divisible by 5 $\therefore$ not divisible by 30.

Similarly true for $n = 5m - 2$.

CONTRADICTION.

$\therefore n(n+1)(n-1)$ not divisible by 30 if $n \equiv 5m \pm 2$, $m \in \mathbb{Z}$.

QED!

d) There exists a student who studies with ATAR < 90.

$\equiv$ At least 1 student who studies has ATAR < 90.

Negation: There are no students who study with ATAR < 90.

OR All students who study for exams have ATAR $\geq$ 90.

(or ATAR $\not<$ 90.) 2

Q12

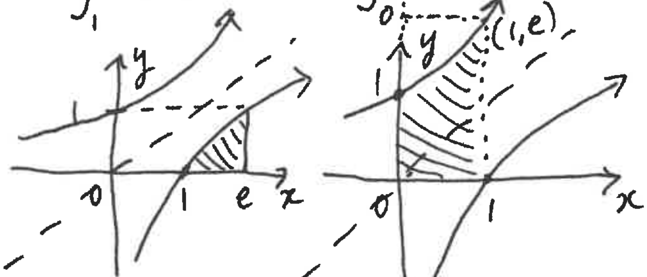
a) $\int \tan x \ln(\cos x)\,dx$

$= -\int -\frac{\sin x}{\cos x} \ln(\cos x)\,dx$

$= \int f'(x) \cdot [f(x)]^1\,dx$

$= -\frac{(\ln(\cos x))^2}{2}$ (2)

b) $\int_1^e \ln x\,dx + \int_0^1 e^x\,dx = e$



Graphically since $y = \ln x$ and $y = e^x$ are inverse functions the two areas form a rectangle $1 \times e = e$ units^2. (2)

c) $z = \cos\theta + i\sin\theta$.

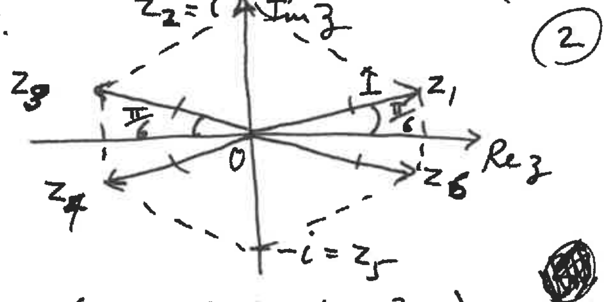
i) $z^6 = -1 = 1\operatorname{cis}\pi$. Spacing $\frac{2\pi}{6} = \frac{\pi}{3}$.

$\therefore$ let $z_1 = \operatorname{cis}\frac{\pi}{6}$, $\therefore z_2 = \operatorname{cis}\left(\frac{\pi}{6} + \frac{\pi}{3}\right)$

$z_3 = \operatorname{cis}\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)$, $z_4 = \operatorname{cis}\left(\frac{\pi}{6} + \pi\right)$,

$z_5 = \operatorname{cis}\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)$, $z_6 = \operatorname{cis}\left(\frac{\pi}{6} - \frac{\pi}{3}\right)$

$\therefore$ (2)

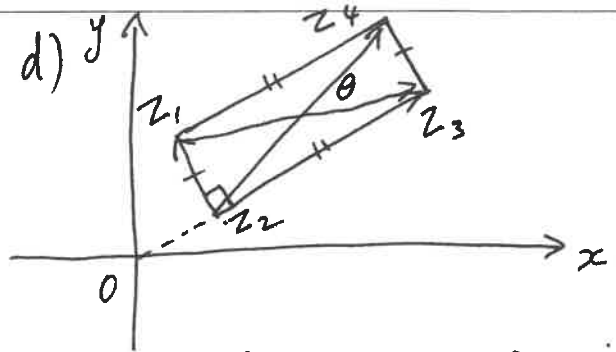


ii) $z^6 + 1 = (z^2+1)(z^4 - z^2 + 1)$

$= (z-i)(z+i)(z^4 - z^2 + 1)$

$\therefore$ Roots of $z^4 - z^2 + 1 = 0$ are z_1, z_3, z_4 and z_6. (2)

d)



i) $z_2 = \sqrt{3} + i$ $\quad z_3 = 4\sqrt{3} + 4i = 4(\sqrt{3} + i)$

We know $2Z_1Z_2 = Z_2Z_3$

Now $|z_1 - z_2| = \frac{1}{2}|z_3 - z_2|$

and $\arg(z_1 - z_2) = \arg(z_3 - z_2) + \frac{\pi}{2}$

$\therefore z_1 - z_2 = \frac{1}{2} i \times (z_3 - z_2)$

$\therefore z_1 = \frac{1}{2} i (z_3 - z_2) + z_2$ (2)

$= \frac{1}{2} i (3\sqrt{3} - 3i) + \sqrt{3} + i$

$= \frac{3i\sqrt{3}}{2} + \frac{3}{2} + \sqrt{3} + i$

$= \left(\sqrt{3} + \frac{3}{2}\right) + i\left(1 + \frac{3\sqrt{3}}{2}\right)$

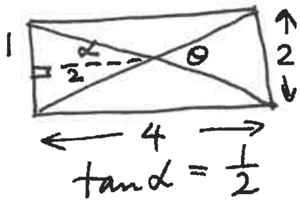
ii) $z_4 = z_1 - z_2 + z_3$

$= \cancel{\sqrt{3}} + \frac{3}{2} + \cancel{i} + \frac{3i\sqrt{3}}{2} + 3\sqrt{3} + 3i$

$= \left(\frac{3}{2} + 4\sqrt{3}\right) + i\left(\frac{3\sqrt{3}}{2} + 4\right)$ (2)

iii) $\arg\left(\frac{z_4 - z_2}{z_3 - z_1}\right) = \arg(z_4 - z_2) - \arg(z_3 - z_1)$

$= \theta$ in diagram.



$\tan\alpha = \frac{1}{2}$

Ratio 2:1

let $\tan\theta = \tan 2\alpha$

$\tan\theta = \frac{2\tan\alpha}{1 - \tan^2\alpha}$

$= \frac{2\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)^2}$ (3)

$= \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$

$\therefore \tan^{-1}\frac{4}{3} = \theta$

Q13

a) $a>0, b>0, s>0$, $a+b=s$

i) RTP: $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{s} = \frac{4}{a+b}$

Proof: Consider the difference,

$$\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b} = \frac{b(a+b)+a(a+b)-4ab}{ab(a+b)}$$

$$= \frac{ab+b^2+a^2+ab-4ab}{ab(a+b)}$$

$$= \frac{a^2-2ab+b^2}{ab(a+b)} \quad (2)$$

$$= \frac{(a-b)^2}{ab(a+b)} \geq 0$$

Since $(a-b)^2 \geq 0$, $a, b > 0$.

$\therefore \frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b} \left(=\frac{4}{s}\right)$ QED.

ii) RTP: $\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{8}{s^2} = \frac{8}{(a+b)^2}$

Proof: Consider:

$$\frac{1}{a^2}+\frac{1}{b^2} \geq \frac{2}{ab}$$

From (i) $\frac{1}{a}+\frac{1}{b} \geq \frac{4}{a+b}$ (2)

$$\therefore \frac{b+a}{ab} \geq \frac{4}{a+b}$$

$$\therefore \frac{1}{ab} \geq \frac{4}{(a+b)^2}$$

$$\therefore \frac{2}{ab} \geq \frac{8}{(a+b)^2}$$

$$\therefore \frac{1}{a^2}+\frac{1}{b^2} \geq \frac{2}{ab} \geq \frac{8}{(a+b)^2}$$

$\therefore \frac{1}{a^2}+\frac{1}{b^2} \geq \frac{8}{s^2}$ QED

b) $P(3,2,1)$ lies on S_1, $C(2,3,4)$

i) PQ is diameter

$\therefore \overrightarrow{PC} = \overrightarrow{CQ}$

$\therefore Q(a,b,c) = (+1,4,7)$ (1)

b) Cont'd:

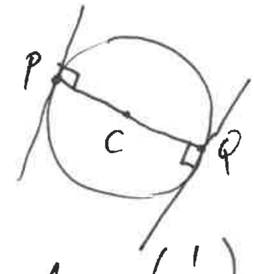
ii) $\underset{\sim}{r} = \begin{pmatrix}3\\2\\1\end{pmatrix} + \lambda\begin{pmatrix}4\\1\\1\end{pmatrix}$. $P(3,2,1)$ lies on S_1. Check directions:

$$\begin{pmatrix}4\\1\\1\end{pmatrix}\cdot\overrightarrow{PC} = \begin{pmatrix}4\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\3\end{pmatrix} \quad (2)$$

$$= 4\times(-1)+1\times1+1\times3$$

$= 0 \quad \therefore \underset{\sim}{r}$ is tangent.

iii)



$\underset{\sim}{l} \perp \underset{\sim}{r}$

$\overrightarrow{PQ} \perp \underset{\sim}{l}$ (3)

let $\underset{\sim}{l} = \begin{pmatrix}1\\4\\7\end{pmatrix} + \mu\begin{pmatrix}a\\b\\c\end{pmatrix}$

So $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}4\\1\\1\end{pmatrix} = 0$ AND $\begin{pmatrix}a\\b\\c\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\3\end{pmatrix} = 0$

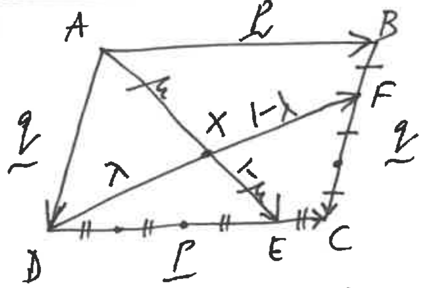
$\therefore 4a+b+c=0$ (1) and $-a+b+3c=0$ (2)

(1) − (2) $\quad 5a-2c=0 \quad \therefore 2c=5a$

let $c=5$ and $a=2$ $\therefore b=-13$

$$\therefore \underset{\sim}{l} = \begin{pmatrix}1\\4\\7\end{pmatrix} + \mu\begin{pmatrix}2\\-13\\5\end{pmatrix}.$$

c)



i) RTP: $\overrightarrow{AX} = \mu\left(\underset{\sim}{q}+\frac{3}{4}\underset{\sim}{p}\right)$

Proof: $\overrightarrow{AX} = \mu\overrightarrow{AE}$

$$= \mu\left(\overrightarrow{AD}+\frac{3}{4}\overrightarrow{DC}\right) \quad (2)$$

$$= \mu\left(\underset{\sim}{q}+\frac{3}{4}\underset{\sim}{p}\right) \quad \text{QED!}$$

[NB: $\overrightarrow{AB} = \overrightarrow{DC}$ since opposite sides of parallelogram equal and parallel.]

Q 13 cont'd

c) cont'd

ii) RTP: $\overrightarrow{AE}$ bisects DF.
i.e. $\overrightarrow{DX} = \overrightarrow{XF}$ or $\lambda = \frac{1}{2}$

Proof: From (i) $\overrightarrow{AX} = \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right)$

$\overrightarrow{AX} = \underset{\sim}{q} + \overrightarrow{DX}$

$\therefore -\overrightarrow{DX} = \underset{\sim}{q} - \overrightarrow{AX}$

$= \underset{\sim}{q} - \mu\left(\underset{\sim}{q} + \frac{3}{4}\underset{\sim}{p}\right)$

$= \underset{\sim}{q} - \mu\underset{\sim}{q} - \frac{3}{4}\mu\underset{\sim}{p}$

$\overrightarrow{XD} = \underset{\sim}{q}(1-\mu) - \frac{3}{4}\mu\underset{\sim}{p}$ ①

Also, $\cancel{\overrightarrow{DX} + \overrightarrow{XE} = \overrightarrow{DE}}$

$\cancel{\overrightarrow{DX} = \overrightarrow{DE} - \overrightarrow{XE}}$

$\cancel{= \frac{3}{4}\overrightarrow{DC} - (1-\mu)\overrightarrow{AE}}$

$\cancel{= \frac{3}{4}\underset{\sim}{p} -}$

$\overrightarrow{DX} = \lambda\overrightarrow{DF}$

$= \lambda\left(\overrightarrow{DC} + \overrightarrow{CF}\right)$

$= \lambda\left(\underset{\sim}{p} + \frac{2}{3}\overrightarrow{CB}\right)$

$= \lambda\left(\underset{\sim}{p} - \frac{2}{3}\underset{\sim}{q}\right)$

$= \lambda\underset{\sim}{p} - \frac{2}{3}\lambda\underset{\sim}{q}$ ②

Equating −① & ②

$\underset{\sim}{q}(1-\mu) - \frac{3}{4}\mu\underset{\sim}{p} = -\lambda\underset{\sim}{p} + \frac{2}{3}\lambda\underset{\sim}{q}$

$\underset{\sim}{q}$ & $\underset{\sim}{p}$ not parallel (independent)

$\therefore$ Equating $\underset{\sim}{q}$s & $\underset{\sim}{p}$s

$\underset{\sim}{q}(1-\mu) = +\frac{2}{3}\lambda\underset{\sim}{q}$

c) ii) Cont'd:

and $-\frac{3}{4}\mu\underset{\sim}{p} = -\lambda\underset{\sim}{p}$

$\therefore \lambda = +\frac{3}{4}\mu$

and $1-\mu = +\frac{2}{3}\lambda$

$\therefore 1-\mu = +\frac{2}{3}\left(+\frac{3}{4}\mu\right)$

$1-\mu = +\frac{1}{2}\mu$

$\therefore 1 = \frac{3}{2}\mu$

$\mu = \frac{2}{3}$. ③

$\therefore \lambda = +\frac{3}{4} \times \frac{2}{3}$

$= +\frac{1}{2}$

$\therefore \overrightarrow{DX} = \frac{1}{2}\overrightarrow{DF}$ X is midpoint

$\therefore \overrightarrow{AE}$ bisects DF.

Q14

a) i) $f(x) = x - n\ln x$, $x > 0$, $n = 1, 2, 3, \ldots$

$f'(x) = 1 - \frac{n}{x}$

$0 = 1 - \frac{n}{x} \Rightarrow x = n.$ (2)

$f''(x) = 0 + nx^{-2} = \frac{n}{x^2}$

When $x = n$ $f''(n) = \frac{n}{n^2} = \frac{1}{n} > 0$

$\therefore$ Min $f(x)$ when $x = n$.

$\therefore$ $f(n) = n - n\ln n$ is the min value.

ii) RTP: $\frac{e^x}{x^n} > \frac{e^n}{n^n}$ $\forall x > n$

Proof: Min value of $f(x)$ occurs at $f(n) = n - n\ln n$.

$\therefore$ $f(x) \geq f(n)$

If $x > n$ then $f(x) > f(n)$

$\therefore$ $x - n\ln x > n - n\ln n$

$n\ln n - n\ln x > n - x$

$\ln n^n - \ln x^n > n - x$

$\therefore$ $\ln\left(\frac{n^n}{x^n}\right) > n - x$

$\frac{n^n}{x^n} > e^{n-x}$

$\therefore$ $\frac{n^n}{x^n} > \frac{e^n}{e^x}$ (2)

or $\frac{e^x}{x^n} > \frac{e^n}{n^n}$. QED

b) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$ $n = 1, 2, 3 \ldots$

RTP: $I_n = \frac{1}{n}\left[\cos^{n-1}x \sin x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n} I_{n-2}$.

b) i) Cont'd: let $dv = \cos x \, dx$

$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \, dx = \int_0^{\frac{\pi}{2}} \cos x . \cos^{n-1} x \, dx$

$= uv - \int v \, du$

$v = \sin x$

$u = \cos^{n-1} x$

$du = (n-1)\cos^{n-2} x \times (-\sin x) dx$

$= \left[\cos^{n-1} x . \sin x\right]_0^{\frac{\pi}{2}}$

$+ (n-1)\int_0^{\frac{\pi}{2}} \sin x \, dx . \cos^{n-2} x . \sin x$

$= \left[\cos^{n-1} x . \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x . \sin^2 x \, dx$

$= \left[\cos^{n-1} x . \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} (1 - \cos^2 x)\cos^{n-2} x \, dx$

$= \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \, dx$

$+ (n-1)\int_0^{\frac{\pi}{2}} -\cos^n x \, dx$

$\therefore I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}} + (n-1) I_{n-2}$

$- (n-1) I_n$

$\therefore I_n + (n-1) I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}}$

$+ (n-1) I_{n-2}$

$\therefore I_n \left(1 + (n-1)\right) = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}}$

$+ (n-1) I_{n-2}$

$\therefore n I_n = \left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}}$

$+ (n-1) I_{n-2}$

$\therefore I_n = \frac{1}{n}\left[\cos^{n-1} x \sin x\right]_0^{\frac{\pi}{2}}$

$+ \frac{(n-1)}{n} I_{n-2}$

(3)

Q14 cont'd:

b) ii) $\int_0^{\frac{\pi}{2}} \cos^5 x \, dx \qquad n=5$

$= \frac{1}{5}\left[\cos^{5-1}x \sin x\right]_0^{\frac{\pi}{2}} + \frac{4}{5}\int_0^{\frac{\pi}{2}} \cos^3 x \, dx$

$= \frac{1}{5}\left[0-0\right] + \frac{4}{5} I_3$

$= \frac{4}{5}\left[\frac{1}{3}\left[\cos^2 x \sin x\right]_0^{\frac{\pi}{2}} + \frac{2}{3}\int_0^{\frac{\pi}{2}} \cos x \, dx\right]$

$= \frac{4}{5}\left[0 + \frac{2}{3}\left[\sin x\right]_0^{\frac{\pi}{2}}\right]$

$= \frac{4}{5} \times \frac{2}{3}\left[1-0\right]$ (2)

$= \frac{8}{15}$.

c) i) RTP: $n! > 2^n$ for $n > 3$.

Proof: Let $P(n)$ be the proposition that $n! > 2^n$ for $n = 4, 5, 6, \ldots$

Prove $P(4)$ ~~true~~:

LHS $= 4! = 24$

RHS $= 2^4 = 16$

$\therefore$ LHS $>$ RHS $P(4)$ true.

Assume $P(k)$ true for some $k \in \mathbb{N}$, $k \geq 4$ ie $k! > 2^k$.

RTP: $P(k+1)$ true ie $(k+1)! > 2^{k+1}$.

Proof: Consider the LHS of $P(k+1)$:

(3)

$(k+1)! = (k+1)k!$

$= k.k! + k!$

$> k.2^k + 2^k$

$> 2^k + 2^k$ since $k \geq 4$

$= 2^{k+1}$ $\therefore P(k+1)$ true.

c) i) cont'd:

$\therefore P(n)$ true $\forall n > 3$ by Math Induction.

ii) RTP: $\sum_4^n \frac{1}{k!} < \sum_4^n \frac{1}{2^k}$

ie $\frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots + \frac{1}{n!}$

$< \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \ldots + \frac{1}{2^n}$

Proof: From (i) $n! > 2^n > 0$ for $n = 4, 5, 6, \ldots$

$\therefore 4! > 2^4 \Rightarrow \frac{1}{4!} < \frac{1}{2^4}$

$5! > 2^5 \Rightarrow \frac{1}{5!} < \frac{1}{2^5}$

$\vdots$

$n! > 2^n \Rightarrow \frac{1}{n!} < \frac{1}{2^n}$

$\therefore$ Adding LHS & RHS:

(2)

$\frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots + \frac{1}{n!}$

$< \frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \ldots + \frac{1}{2^n}$

QED!

iii) As $n \to \infty$,

$\frac{1}{2^4} + \frac{1}{2^5} + \frac{1}{2^6} + \ldots \to \frac{a}{1-r}$

since $|r| < 1$, $r = \frac{1}{2}$, $a = \frac{1}{2^4}$

$\therefore S_\infty = \frac{\frac{1}{2^4}}{1-\frac{1}{2}} = \frac{1}{8}$

(1)

But $\sum_4^n \frac{1}{k!} < \sum_4^n \frac{1}{2^k}$

$\therefore \lim_{n\to\infty} \sum_4^n \frac{1}{k!} < \frac{1}{8}$

$\therefore \lim_{n\to\infty} \left(\frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \ldots + \frac{1}{n!}\right) < \frac{1}{8}$

Q15

a) i) RTP: $\tan^{-1}(k+1) - \tan^{-1}k = \tan^{-1}\left(\frac{1}{k(k+1)+1}\right)$ for $k=1,2,3,\dots$

Proof: Consider $\tan^{-1}(k+1) = \alpha$

$\tan^{-1}(k) = \beta$

then $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

2 $\therefore \quad = \dfrac{k+1-\cancel{k}}{1+(k+1)k}$

QED!

$\therefore \alpha - \beta = \tan^{-1}\left(\dfrac{1}{k(k+1)+1}\right)$

ii) RTP: $\lim\limits_{n\to\infty}\left(\tan^{-1}\left(\frac{1}{1.2+1}\right) + \tan^{-1}\left(\frac{1}{2.3+1}\right) + \dots + \tan^{-1}\left(\frac{1}{n(n+1)+1}\right)\right) = \frac{\pi}{4}$.

Proof: Consider LHS:

$\lim\limits_{n\to\infty}\left(\tan^{-1}\left(\frac{1}{1.2+1}\right) + \tan^{-1}\left(\frac{1}{2.3+1}\right) + \dots + \tan^{-1}\left(\frac{1}{n(n+1)+1}\right)\right)$

3

$= \lim\limits_{n\to\infty}\big(\cancel{\tan^{-1}2} - \tan^{-1}1$

$+ \cancel{\tan^{-1}3} - \cancel{\tan^{-1}2} + \cancel{\tan^{-1}4} - \cancel{\tan^{-1}3}$

$+ \dots + \cancel{\tan^{-1}n} - \cancel{\tan^{-1}(n-1)}$

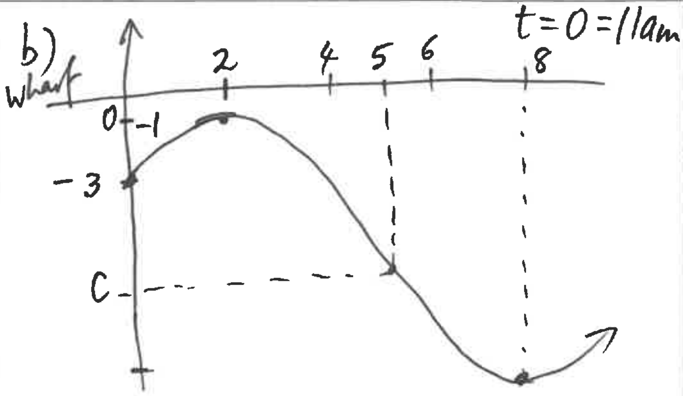
$+ \tan^{-1}(n+1) - \cancel{\tan^{-1}n}\big)$

$= \lim\limits_{n\to\infty}\left(\tan^{-1}(n+1) - \tan^{-1}1\right)$

$= \frac{\pi}{2} - \frac{\pi}{4}$

$= \frac{\pi}{4}$.

b) $t=0=11am$



Assume SHM ie.

$x = A\sin(nt+\varepsilon) + C$

$\dot{x} = An\cos(nt+\varepsilon)$

$\ddot{x} = -An^2\sin(nt+\varepsilon)$

$\therefore \ddot{x} = -n^2 x$

i) Period $= \frac{2\pi}{n}$ High tide from Low tide is 6h

$\therefore 12h = \frac{2\pi}{n}$ $\quad\therefore$ Period $= 12h$

2

$n = \frac{2\pi}{12} = \frac{\pi}{6}$

$\therefore x = A\sin\left(\frac{\pi}{6}t + \varepsilon\right) + C$

ii) When $t=0$, $x=-3$

$\therefore -3 = A\sin(0+\varepsilon) + C$

$-3 = A\sin\varepsilon + C$

When $t=2$, $x=-1$, $v=0$

$-1 = A\sin\left(\frac{\pi}{3}+\varepsilon\right) + C$

$0 = \frac{A\pi}{6}\cos\left(\frac{2\pi}{6}+\varepsilon\right)$

3

$\cos\left(\frac{\pi}{3}+\varepsilon\right) = 0$

$\therefore \frac{\pi}{3}+\varepsilon = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$

$\therefore \varepsilon = \frac{\pi}{6}, \dots$

$\therefore x = A\sin\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) + C$

$\dot{x} = \frac{A\pi}{6}\cos\left(\frac{\pi}{6}t + \frac{\pi}{6}\right)$

Q15 cont'd

b) cont'd ii) Cont'd:

When $t=2$ $\quad -1 = A\sin\left(\frac{\pi}{2}\right) + C$

$-1 = A + C \quad ①$

~~$t=8 \quad C-A = A\sin\left(\frac{4\pi}{3}+\frac{\pi}{6}\right)+C$~~

~~Hmm! $\quad C-A = -A+C$~~

~~$t=5 \quad C = A\sin(\pi) + C$~~

~~Hmm!~~

$t=0, \quad x = -3 = A\sin\frac{\pi}{6} + C$

$-3 = \frac{A}{2} + C \quad ②$

$① - ② \qquad +2 = \frac{A}{2}$

$A = 4, C = -5 \quad 3$

$\therefore \quad x = 4\sin\left(\frac{\pi}{6}t + \frac{\pi}{6}\right) - 5$

$\therefore$ Amplitude $= 4$.

iii) Tide rising fastest between high & low tides (Max velocity)

$= 8 + 3$

$= 11$ hours after 11am $\quad 2$

$= 10$pm.

c) $(1+i)^n = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n$

$= \sqrt{2}^n\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$

Also

$(1+i)^n = {}^nC_0 1^n i^0 + {}^nC_1 i^1 + {}^nC_2 i^2 + {}^nC_3 i^3$

$+ {}^nC_4 i^4 + {}^nC_5 i^5 + \ldots + {}^nC_{n-1} i^{n-1}$

$+ {}^nC_n i^n$

Q15 cont'd:

c) cont'd

$\therefore (1+i)^n = \binom{n}{0} + \binom{n}{1}i - \binom{n}{2} - \binom{n}{3}i$

$+ \binom{n}{4} + \binom{n}{5}i - \binom{n}{6} - \binom{n}{7}i + \binom{n}{8}$

$+ \ldots$

Equating Reals:

$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \ldots = \sqrt{2}^n \cos\frac{n\pi}{4}$

$= 2^{\frac{n}{2}} \cos\frac{n\pi}{4}$.

3

Solutions to
Title: Ascham 2024 Trial Math Ext 2

Q16

a) $\int \frac{\cos x \, dx}{1+\cos x} = \int \frac{\cos x + 1 - 1 \, dx}{1+\cos x}$

$= \int \frac{1+\cos x}{1+\cos x} - \frac{1}{1+\cos x} \, dx$

$= \int 1 \, dx - \int \frac{1}{1+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2}$

$= x - \int \frac{1}{1+\cancel{t^2}+1-\cancel{t^2}} \times 2dt$

3

$= x - \int \frac{1}{\cancel{2}} \times \cancel{2}\, dt$

let $t = \tan\frac{x}{2}$, $dx = \frac{2dt}{1+t^2}$

$= x - t + C$

$= x - \tan\frac{x}{2} + C$

b) i) RTP: $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

Proof: $\cos 4\theta = \cos(2\theta + 2\theta)$

$= \cos 2\theta \cos 2\theta - \sin 2\theta \sin 2\theta$

3

$= (2\cos^2\theta - 1)^2 - (2\sin\theta\cos\theta)^2$

$= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\sin^2\theta\cos^2\theta$

$= 4\cos^4\theta - 4\cos^2\theta + 1 - 4(1-\cos^2\theta)\cos^2\theta$

$= 4\cos^4\theta - 4\cos^2\theta + 1 - 4\cos^2\theta + 4\cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ QED!

ii) $16x^4 - 16x^2 + 1 = 0$

$16x^4 - 16x^2 = -1$

$8x^4 - 8x^2 = -\frac{1}{2}$

let $x = \cos\theta$

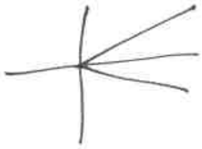
then $8x^4 - 8x^2 + 1 = \cos 4\theta$

then $8x^4 - 8x^2 = \cos 4\theta - 1$

$\therefore -\frac{1}{2} = \cos 4\theta - 1$

$\therefore \cos 4\theta = \frac{1}{2}$

b) ii) Cont'd:



Solve $\cos 4\theta = \frac{1}{2}$

$4\theta = \frac{\pi}{3}, -\frac{\pi}{3}, \pm 2\pi + \frac{\pi}{3}, \pm 2\pi - \frac{\pi}{3}, \pm 4\pi + \frac{\pi}{3}, \pm 4\pi - \frac{\pi}{3}, \pm 6\pi + \frac{\pi}{-3}, \ldots$

$= \frac{\pi}{3}, \frac{-\pi}{3}, \frac{7\pi}{3}, \frac{5\pi}{3}, \frac{13\pi}{3}, \frac{11\pi}{3}, \frac{19\pi}{3}, \frac{17\pi}{3}, \ldots$

$\theta = \frac{\pi}{12}, \frac{-\pi}{12}, \frac{7\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{17\pi}{12}, \ldots$ 2

$\therefore$ Solutions to $16x^4 - 16x^2 + 1 = 0$ are $\cos\frac{\pi}{12}, \cos\left(\frac{-\pi}{12}\right), \cos\frac{7\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{13\pi}{12}, \cos\frac{11\pi}{12}, \cos\frac{19\pi}{12}, \cos\frac{17\pi}{12}, \ldots$ etc. and the negatives $\left(-2\pi \pm \frac{\pi}{3} \text{ etc}\right)$

So 4 unique solutions are $\cos\frac{\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{11\pi}{12}$.

iii) Exact values will be roots of $16x^4 - 16x^2 + 1 = 0$ 2

$x^2 = \frac{16 \pm \sqrt{16^2 - 4(16)(1)}}{2(16)}$

$= \frac{16 \pm \sqrt{192}}{32}$

$= \frac{16 \pm 8\sqrt{3}}{32}$

$= \frac{\cancel{8}(2 \pm \sqrt{3})}{\cancel{32}}$

$= \frac{2 \pm \sqrt{3}}{4}$

$\therefore x = \pm\sqrt{\frac{2 \pm \sqrt{3}}{4}}$

$= \pm\frac{\sqrt{2 \pm \sqrt{3}}}{2}$

Since $\cos\frac{5\pi}{12} < \cos\frac{\pi}{12}$ but both are > 0 then $\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$

and $\cos\frac{5\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}$.

Q16 cont'd

c) $x = 2v - \tan^{-1} v$ $t=0$, $x=0$, $v=1$

i) RTP: $v\dfrac{dv}{dx} = \dfrac{v(1+v^2)}{1+2v^2}$

Proof: $x = 2v - \tan^{-1} v$

$\therefore 1\,dx = 2dv - \dfrac{1}{1+v^2}dv$

$dx = \left(2 - \dfrac{1}{1+v^2}\right)dv$

$\therefore \dfrac{dx}{dv} = \dfrac{2+2v^2-1}{1+v^2}$

$\therefore \dfrac{dv}{dx} = \dfrac{1+v^2}{1+2v^2}$ 2

$\therefore v\dfrac{dv}{dx} = \dfrac{v(1+v^2)}{1+2v^2}$

ii) $t = ?$ when $v = 7$.

$\ddot{x} = \dfrac{v\,dv}{dx}$

$\therefore \dfrac{dv}{dt} = \dfrac{v(1+v^2)}{1+2v^2}$

$\displaystyle\int_0^7 \frac{1+2v^2}{v(1+v^2)} = \int_0^T dt$

let $\dfrac{1+2v^2}{v(1+v^2)} = \dfrac{A}{v} + \dfrac{Bv+C}{1+v^2}$

$\therefore 1+2v^2 \equiv A(1+v^2) + v(Bv+C)$

$v=0 \quad 1 = A$

$\therefore 1+2v^2 = 1+v^2+v(Bv+C)$

$v=1 \quad 1+2 = 1+1+1(B+C)$

$1 = B+C$ ①

$v=-1 \quad 3 = 2-1(C-B)$

$1 = B-C$ ②

16 cont'd

c) ii) cont'd:

① + ② $2 = 2B$

$B = 1 \quad \therefore C = 0$

$\therefore T = \displaystyle\int_0^7 \frac{1}{v} + \frac{v}{1+v^2}\,dv$

$= \left[\ln v + \frac{1}{2}\ln|1+v^2|\right]_1^7$

$= \ln 7 + \frac{1}{2}\ln 50 - \ln 1 - \frac{1}{2}\ln 2$

$= \ln 7 + \frac{1}{2}\ln 25$

$= \ln 7 + \ln 5$ 3

~~$= \ln \frac{7}{125} < 0$ Hmm.~~

$= \ln 35.$